DETERMINATION OF OPTIMAL CONJUGATE STRESS STRAIN PAIRS

Bohuslav Stříž and Lukáš Čapek

Technical University of Liberec, Faculty of Textile Engineering, Department of Technologies and Structures Studentská 2, 461 17 Liberec, Czech Republic
bohuslav.striz@tul.cz; lukas.capek@tul.cz

Abstract: *The aim of this paper was to shown a new methodology for determining and optimal conjugated pair for Cauchy stress tensor. Methodology demonstrated in this work can be used for various materials loaded by uniaxial tension, e.g. for textiles, soft tissues etc.*

Keywords: *Stress tensor, strain tensor, conjugated pair.*

1 INTRODUCTION

In engineering practice, the mechanical parameters are usually identified in a simplified form as engineering stress and deformation. These material parameters are used worldwide since 19th century for linear tasks. When large deformation presents, as in composite materials or others, this simplified approach cannot be used any more. It is necessary to define these deformations and stress as second order tensors that are energetically conjugated. It means that their double-dot product express strain energy in the system.

When expressing derivate quantity, e.g. Young moduli, according to different conjugated pairs, different values will be obtained [1, 2]. In view of the fact that the trues stress tensor is not conjugated with any known strain tensor, it is difficult to decide about suitable conjugated pair. It will be shown, that for uniaxial loading of the specimen it is possible to determine to Cauchy stress tensor a suitable conjugated strain tensor.

2 MATERIALS AND METHODS

Let we have a specimen of an anisotropic material, loaded by a force F_1. Cartesian coordinates of points 1-4 were determined by an optical method.

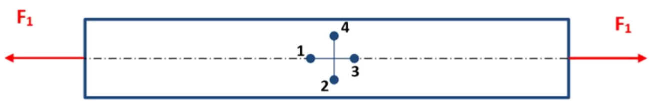

Figure 1 Testing sample

From Cartesian coordinates of points 1-4 it is possible to calculate displacements as:

$$\begin{aligned}
(1+u_{11})(x_1^{01}-x_1^{03})+u_{12}(x_2^{01}-x_2^{03}) &= x_1^2-x_1^3\\
(1+u_{11})(x_1^{02}-x_1^{04})+u_{12}(x_2^{02}-x_2^{04}) &= x_1^2-x_1^4\\
u_{21}(x_1^{01}-x_1^{03})+(1+u_{22})(x_2^{01}-x_2^{03}) &= x_2^1-x_2^3\\
u_{21}(x_1^{02}-x_1^{04})+(1+u_{22})(x_2^{02}-x_2^{04}) &= x_2^2-x_2^4
\end{aligned} \qquad (1)$$

where the Langragarian coordinate system is noticed as x_i^{0j} and Euler´s coordinate system is x_i^j.

For uniaxial type of loading (symmetrical sample) the equations (1) can be simplified. Displacements u_{12} and u_{21} are zero. Thus:

$$\begin{aligned}
(1+u_{11})(x_1^{01}-x_1^{03}) &= x_1^1-x_1^3\\
(1+u_{22})(x_1^{02}-x_1^{04}) &= x_2^2-x_2^4
\end{aligned} \qquad (2)$$

If we assume the lines 13 and 24 perpendicular, than only two equations can be used. For lower values of parameter x the higher scatter of coordinates x_i^j is. So that the scatter of displacement u_{11} and u_{22} is also higher.

Regarding the fact that the sample is slightly press-stress at the beginning of the measurement so that the displacement u_{11}, u_{22} will not be zero at the beginning of measurement too. Let´s suppose that displacement evaluates linearly with some constant k_1 and k_2. We can write it down as:

$$\begin{aligned}
u_{11} &= u_{11}^0+k_1x\\
u_{22} &= u_{22}^0+k_2x
\end{aligned} \qquad (3)$$

It is clear that $u_{11}^0 > 0$, $k_1 > 0$, $u_{22}^0 < 0$, $k_2 < 0$, because the sample is lengthen along line 13 and shorten along the line 24.

Suppose we have a variable h as an actual thickness of the sample and h_0 as an initial thickness.

We can write the transversal displacement as:

$$\frac{h}{h_0} - 1 = u_{33} \tag{4}$$

where $u_{33} \leq 0$ due the thinning of the sample.

The deformation gradient F is then:

$$F = \begin{pmatrix} 1+u_{11} & 0 & 0 \\ 0 & 1+u_{22} & 0 \\ 0 & 0 & 1+u_{33} \end{pmatrix} \tag{5}$$

If we assume initial length of sample l_0, the Cauchy stress can be expressed by:

$$S_{11} = \frac{F_1}{l_0 h_0 (1+u_{22})(1+u_{33})} \tag{6}$$

which forms the first member of Cauchy stress tensor:

$$\sum = \begin{pmatrix} S_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7}$$

Further we can write Biot´s stress tensor:

$$S_B = \frac{J}{2}\left[F^{-1}\sum R + R^T \sum F^{T-1}\right] \tag{8}$$

where R is a rotational matrix, J is a Jacobian and F a deformation gradient.

In case of uniaxial loading, we simply write an inversion of deformation gradient as inversion of diagonal matrix:

$$F^{-1} = \begin{pmatrix} \frac{1}{1+u_{11}} & 0 & 0 \\ 0 & \frac{1}{1+u_{22}} & 0 \\ 0 & 0 & \frac{1}{1+u_{33}} \end{pmatrix} = F^{T-1} \tag{9}$$

After some manipulating with expressions, we finally get a formulation of Biot's stress tensor for uniaxial loading:

$$S_{\boldsymbol{B}} = \frac{(1+u_{11})(1+u_{22})(1+u_{33})}{2} \cdot \\ \cdot \left[\frac{F_1}{l_0 h_0 (1+u_{11})(1+u_{22})(1+u_{33})}\right] 2 = \\ = \frac{F_1}{l_0 h_0} = C_{11} \tag{10}$$

$$S_B = \begin{pmatrix} C_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11}$$

Stress tensor for conjugated pairs

In order to get a conjugated pair of stress-strain tensors, we define a exponent-generalization of different stress tensors as:

m = 2 Piola-Kirchov
m = 1 Biott
m = 0 logarithmic
m = -1 Cernych
m = -2 Hill, Almansi

$$S(m) = \frac{1}{2}\left[S_B U^{1-m} + U^{1-m} S_B\right] \tag{12}$$

Since we know all members of above equation we can simply write down a formulation for uniaxial stress tensor in a generalized form:

$$S(m) = \begin{pmatrix} C_{11}(1+u_{11})^{1-m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

this expression is valid for Cauchy stress tensor, but with unknown parameter m. The strain tensor that fulfill the term of conjugation is defined by:

$$\varepsilon_{ij}(m) = \frac{1}{m}\left[U^m - I\right] \tag{14}$$

For above mentioned parameters m, it´s necessary to find out power of strain tensor U, except of m=0. In this case numerator and denominator are zero (14). Calculating limit following equation is obtained:

$$\varepsilon_{ij}(0) = \ln U \tag{15}$$

Equation (15) is valid only in the case when augmenting deformation the main axes of the tensor U is not rotated. When using different values of parameter m we obtained following strain tensors:

$$\varepsilon_{ij}(2) = \frac{1}{2}\left[\begin{pmatrix} (1+u_{11})^2 & 0 & 0 \\ 0 & (1+u_{22})^2 & 0 \\ 0 & 0 & (1+u_{33})^2 \end{pmatrix} - \\ - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \\ = \begin{pmatrix} u_{11} + \frac{1}{2}{u_{11}}^2 & 0 & 0 \\ 0 & u_{22} + \frac{1}{2}{u_{22}}^2 & 0 \\ 0 & 0 & u_{33} + \frac{1}{2}{u_{33}}^2 \end{pmatrix} \tag{16}$$

$$\varepsilon_{ij}(1) = \frac{1}{2}\left[\begin{pmatrix} 1+u_{11} & 0 & 0 \\ 0 & 1+u_{22} & 0 \\ 0 & 0 & 1+u_{33} \end{pmatrix} - \\ - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \\ = \begin{pmatrix} u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & 0 & u_{33} \end{pmatrix} \tag{17}$$

$$\varepsilon_{ij}(0)=\begin{pmatrix}\ln(1+u_{11}) & 0 & 0\\ 0 & \ln(1+u_{22}) & 0\\ 0 & 0 & \ln(1+u_{33})\end{pmatrix} \quad (18)$$

$$\varepsilon_{ij}(-2)=\frac{1}{2}\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}\frac{1}{(1+u_{11})^2}&0&0\\0&\frac{1}{(1+u_{22})^2}&0\\0&0&\frac{1}{(1+u_{33})^2}\end{pmatrix}\right]= \quad (19)$$

$$=\begin{pmatrix}\frac{u_{11}+\frac{1}{2}u_{11}^2}{(1+u_{11})^2}&0&0\\0&\frac{u_{22}+\frac{1}{2}u_{22}^2}{(1+u_{22})^2}&0\\0&0&\frac{u_{33}+\frac{1}{2}u_{33}^2}{(1+u_{33})^2}\end{pmatrix}$$

Similarly, stress tensor *S(m)* might be determined:

$$S(2)=\frac{C_{11}}{1+u_{11}} \quad (20)$$

$$S(1)=C_{11} \quad (21)$$

$$S(0)=C_{11}(1+u_{11}) \quad (22)$$

$$S(-1)=C_{11}(1+u_{11})^2 \quad (23)$$

$$S(-2)=C_{11}(1+u_{11})^3 \quad (24)$$

For Cauchy stress tensor following equations is obtained:

$$S(m)=C_{11}(1+u_{11})^{1-m} \quad (25)$$

$$\varepsilon_{ij}(m)=$$
$$=\frac{1}{m}\left[\begin{pmatrix}(1+u_{11})^m-1&0&0\\0&(1+u_{22})^m-1&0\\0&0&(1+u_{33})^m-1\end{pmatrix}\right] \quad (26)$$

Comparing relation (6) and (25) following equation is obtained

$$\frac{1}{(1+u_{22})(1+u_{33})}=(1+u_{11})^{1-m}$$
$$m=1+\frac{\ln(1+u_{22})+\ln(1+u_{33})}{\ln(1+u_{11})} \quad (27)$$

Parameter *m* is a function of thickness defined by (4). By substituting u_{11}, u_{22}, u_{33} by terms (3) and (4) a functional dependency of parameter *m* on displacement *x* and k_3 is received:

$$m=1+\frac{\ln(1+u^0{}_{22+}k_2x)+\ln(1+k_3x)}{\ln(1+u^0{}_{11+}k_1x)} \quad (28)$$

Equation (28) describes formally Cauchy conjugated pair for uniaxial type of loading. It´s clear that the result is strongly influence by parameter k_3.

The requirement of dot product for equations (25) and (26) will be shown:

$$A=\int_0^{x_{max}}S(m)\frac{d\varepsilon_{11}(m)}{dx}dx=$$
$$=\int_0^{x_{max}}C_{11}(1+u_{11})^{1-m}(1+u_{11})^{m-1}\frac{du_{11}}{dx}dx= \quad (29)$$
$$=\int_0^{x_{max}}\frac{F_1}{l_0h_0}k_1dx$$

Theoretically the parameter “*m*” can reaches any value, but there is only one corresponds to Cauchy stress tensor. According to Hook law:

$$\overline{E_{11}}\varepsilon_{11}(m)+\overline{E_{12}}\varepsilon_{22}(m)\text{-J}C_{11}(1+u_{11})^{1\text{-m}}=0$$
$$\overline{E_{12}}\varepsilon_{11}(m)+\overline{E_{22}}\varepsilon_{22}(m)=0 \quad (30)$$
$$\overline{E_{11}E_{22}}-\overline{E_{12}}^2-\overline{E_4}(\overline{E_{11}}+\overline{E_{22}}-2\overline{E_{12}})=0$$

where:

$$\overline{E}_{12}=\nu(m)\sqrt{\overline{E_{11}E_{22}}}$$
$$\nu(m)=-\frac{\varepsilon_{22}(m)}{\varepsilon_{11}(m)} \quad (31)$$

Modules $\overline{E}_{11}, \overline{E}_{22}$ and $\overline{E}_4$ can be found. Poisson ratio *v(m)* depends on choice of conjugated pair.

An invariant shear modulus $\widetilde{\overline{E}_4}(m)$ is given:

$$\widetilde{\overline{E}_4}(m)=\frac{1}{2}\left[\frac{1}{4}(\overline{E_{11}}+\overline{E_{22}}-2\overline{E_{12}})+\overline{E}_4\right] \quad (32)$$

In engineering practice following expression is often used:

$$G=\frac{\tau_i}{\gamma_i} \quad (33)$$

Where:

$$G=\frac{E}{2(1+\nu)} \quad (34)$$

$$\tau_i=\frac{1}{\sqrt{6}}\sqrt{(S_{11}-S_{22})^2+S_{11}^2+S_{22}^2+6S_{12}^2} \quad (35)$$

$$\gamma_i=\sqrt{\frac{2}{3}}\sqrt{(\varepsilon_{11}-\varepsilon_{22})^2+(\varepsilon_{22}-\varepsilon_{33})^2+(\varepsilon_{33}-\varepsilon_{11})^2+6S_{12}^2} \quad (36)$$

In our case can be found:

$$\widetilde{\overline{E}_4}=\frac{\frac{1}{\sqrt{3}}S_{11}}{\sqrt{\frac{2}{3}}\sqrt{(\varepsilon_{11}-\varepsilon_{22})^2+(\varepsilon_{22}-\varepsilon_{33})^2+(\varepsilon_{33}-\varepsilon_{11})^2}} \quad (37)$$

This makes a system of equations that enables to find out values of “*m*” and u_{33}.

3 CONCLUSION

When evaluating the experimental data of a material, usually it is performed as a uniaxial tension test. What is actually measured is a force versus displacement curve, but in order to make these results independent of specimen size, the results are usually presented as stress versus strain. It is interesting that most, perhaps even all, stress definitions can be paired with a corresponding strain tensor. They come in pairs such that the product of the two will give strain energy. This does not mean that the corresponding pairs must be used together when performing structural analyses. But they must be when computing strain energy density. In view of the fact that the trues stress tensor is not conjugated with any known strain tensor, it is difficult to decide about suitable conjugated pair. It was shown, that for uniaxial loading of the specimen it is possible to determine to Cauchy stress tensor a suitable conjugated strain tensor. Methodology demonstrated in this work can be used for various materials loaded by uniaxial tension, e.g. for textiles, soft tissues, etc.

4 REFERENCES

1. Holzapfel G.A.: Nonlinear Solid Mechanics: A Continuum Approach for Engineering, Wiley, 2000, 470 p., ISBN 0471823198
2. Ogden R.W.: Non-Linear Elastic Deformations, Dover, 1997, 532 p., ISBN 0486696480